

Quantum Scalar Field in D-dimensional de Sitter Spacetimes

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Abstract

In this work we investigate the quantum theory of scalar fields propagating in a D -dimensional de Sitter spacetime. The method of dynamic invariants is used to obtain the solution of the time-dependent Schrödinger equation. The quantum behavior of the scalar field in this background is analyzed, and the results generalize previous ones found in the literature. We point that the Bunch-Davies thermal bath depends on the choice of D and the conformal parameter ξ . This is important in extra dimension physics, as in the Randall-Sundrum model.

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1 Introduction

Although String Theory is a promising solution to the quantization of gravity,[1, 2, 3], in cosmological scales gravity field can be considered as a classical theory and the fields as propagating waves in the background. Gauge and scalar fields, for instance, can be quantized in this background by the use of semiclassical approach [4]. This is very similar to the early days of Quantum Field Theory. The quantization of matter was very established but the quantization of the fields was not well understood. Therefore, many calculations were done considering fields as backgrounds. Time-dependent backgrounds are used to describe many physical systems yielding interesting results. For instance, in the study of black hole evaporation [5], the Unruh, and Casimir effects [6, 7]. They are also very useful to describe the dynamical evolution of the universe, where the production of particles in cosmological spacetimes has been investigated [8, 9, 10].

The core idea of extra dimensional models is to consider the four-dimensional universe as a hyper-surface embedded in a multidimensional manifold. After the proposal of Kaluza and Klein, this idea attracted not much attention. This changed a lot after the advent of supergravity and superstring theory, where extra dimensions are a necessary ingredient. More recently, after the Randall and Sundrum proposal of a Brane world with non factorisable metric there have been an extensive use of these ideas [11, 12]. This model provides a possible solution to the hierarchy problem and show how gravity is trapped to a membrane. After while, this model has been modified to consider the membrane as a topological defect generated by a scalar field [13, 14]. This solves some problems related to the localization of fields in the membrane [15, 16].

In 2004, Carvalho, Furtado, and Pedrosa [17] investigated the quantum scalar fields in a Friedman-Robertson-Walker (FRW) background. They demonstrated that the problem of the field quantization in this background reduces to solve the time-dependent Schrödinger equation for the harmonic oscillator with time-dependent mass and frequency. To solve the time-dependent Schrödinger equation (TDSE), they employed the dynamical method of Lewis and Riesenfeld [18]. By considering a quadratic invariant (I), they found the exact solution of the problem and established the existence of squeezed states in this background.

The quantum effects of a massive scalar field in the de Sitter spacetime was investigated in Ref.[19], where Lopes *et al.* used exact linear invariants and the Lewis and Riesenfeld method to derive the corresponding Schrödinger states in terms of solutions of a second-order ordinary differential equation. They also constructed Gaussian wave packet states and calculate the quantum dispersion as well as the quantum correlations for each mode of the quantized scalar field. Aspects related to Bunch-Davies vacuum for the scalar field using the same method has been analyzed in [20].

The calculations presented in Refs. [19, 20] were performed in a $D = 4$ spacetime. Here we intend to generalize the quantization of the scalar field in a D-dimensional spacetime also by using the Lewis and Riesenfeld method. First, we analyze the scalar quantum field in a D-dimensional FRW background, and, second we obtain the exact solution for a D -dimensional de Sitter spacetime.

2 Decomposition of the Scalar Field

Consider the scalar field in a D -dimensional Friedmann-Robertson-Walker (FRW) spacetime. The Lagrangian density for the scalar field is given by

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\xi R\Phi^2 - \frac{1}{2}\mu\Phi^2, \quad (1)$$

where the metric $ds^2 = -dt^2 + a^2(t)d\vec{x}\cdot d\vec{x}$ and R is the Ricci scalar. The scalar field can be decomposed in a complete basis u_k given by

$$u_k(\vec{x}, t) = e^{i\vec{k}\cdot\vec{x}}\phi_k(t) \equiv e^{i\vec{k}\cdot\vec{x}}\frac{\phi_k^1(t) + i\phi_k^2(t)}{\sqrt{2}}, \quad (2)$$

where $i = 1, 2$ labels the real and imaginary parts of ϕ_k . With the definition $\omega^2 = k^2/a^2 + \mu^2 + \xi R$ the action reads

$$S = \frac{1}{2} \sum_{i=1,2} \int dt \int \frac{d^{D-1}k}{(2\pi)^{D-1}} a^{(D-1)} \left[\dot{\phi}_k^i{}^2 - \omega^2(t)\phi_k^i{}^2 \right]. \quad (3)$$

From the above action we obtain the Hamiltonian for the each mode of the scalar field

$$H_{ik} = \frac{1}{2}(a^{-(D-1)}(t)p_{ik}^2 + \omega^2 a^{(D-1)}(t)q_{ik}^2), \quad (4)$$

where

$$p_{ik} = \frac{\partial L}{\partial \dot{\phi}_{ik}} = a(t)^{(D-1)}\dot{\phi}_{ik}, \quad (5)$$

with p being the conjugate momentum. The classical equation of motion for the q th mode reads

$$\ddot{q}_{\lambda ik} + (D-1)\frac{\dot{a}}{a}\dot{q}_{ik} + \omega^2 q_{ik}^2 = 0. \quad (6)$$

Next, consider the classical harmonic oscillator with time-dependent mass and frequency given by the Hamiltonian

$$H(t) = \frac{p^2}{m(t)} + \frac{1}{2}m(t)\omega^2(t)q^2, \quad (7)$$

where $[q, p] = i\hbar$. The equation of motion reads

$$\ddot{q} + \frac{\dot{m}(t)}{m(t)}\dot{q} + \omega^2(t)q = 0, \quad (8)$$

which is similar to Eq. (6) if one considers that each mode of the scalar field corresponds to the time-dependent harmonic oscillator with $m(t) = a^{D-1}(t)$ and $\omega = (k^2/a^2 + \mu^2 + \xi R)^{\frac{1}{2}}$.

3 Quantization of the Scalar Field with Emarkov Approach

Consider a time-dependent harmonic oscillator described by Eq. (7). It is well known that an invariant for Eq. (7) is given by [26]

$$I = \frac{1}{2} \left[\left(\frac{q}{\rho} \right)^2 + (\rho p - m \dot{\rho} q)^2 \right] \quad (9)$$

where $q(t)$ satisfies Eq.(8) and $\rho(t)$ satisfies the generalized Milne-Pinney equation [22, 23]

$$\ddot{\rho} + \gamma(t)\dot{\rho} + \omega^2(t)\rho = \frac{1}{m^2(t)\rho^3} \quad (10)$$

with $\gamma(t) = \dot{m}(t)/m(t)$. The invariant $I(t)$ satisfies the equation

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{i\hbar} [I, H] = 0 \quad (11)$$

and can be considered hermitian if we choose only the real solutions of Eq. (10). Its eigenfunctions, $\phi_n(q, t)$, are assumed to form a complete orthonormal set with time-independent discrete eigenvalues, $\lambda_n = (n + \frac{1}{2})\hbar$. Thus

$$I\phi_n(q, t) = \lambda_n\phi_n(q, t) \quad (12)$$

with $\langle \phi_n, \phi_{n'} \rangle = \delta_{nn'}$. Taking the Schrödinger equation (SE)

$$i\hbar \frac{\partial \psi(q, t)}{\partial t} = H(t)\psi(q, t) \quad (13)$$

where $H(t)$ is given by Eq. (7) with $p = -i\hbar \frac{\partial}{\partial q}$, Lewis and Riesenfeld [18] showed that the solution $\psi_n(q, t)$ of the SE (see Eq.(13)) is related to the functions $\phi_n(q, t)$ by

$$\psi_n(q, t) = e^{i\theta_n(t)} \phi_n(q, t) \quad (14)$$

where the phase functions $\theta_n(t)$ satisfy the equation

$$\hbar \frac{d\theta_n(t)}{dt} = \left\langle \phi_n(q, t) \left| i\hbar \frac{\partial}{\partial t} - H(t) \right| \phi_n(q, t) \right\rangle. \quad (15)$$

The general solution of the SE may be written as

$$\psi(q, t) = \sum_n c_n e^{i\theta_n(t)} \phi_n(q, t) \quad (16)$$

where c_n are time-independent coefficients. Now, using an unitary transformation and following the steps drawn in Ref. [26] we find

$$\begin{aligned}\psi_n(q, t) = & e^{i\theta_n(t)} \left(\frac{1}{\pi^{1/2} \hbar^{1/2} n! 2^n \rho} \right)^{1/2} \times \\ & \exp \left\{ \frac{im(t)}{2\hbar} \left[\frac{\dot{\rho}}{2\hbar} + \frac{i}{m(t)\rho^2(t)} \right] q^2 \right\} \times \\ & H_n \left(\frac{1}{\sqrt{\hbar}} \frac{q}{\rho} \right)\end{aligned}\quad (17)$$

where

$$\theta_n(t) = -(n + \frac{1}{2}) \int_{t_0}^t \frac{1}{m(t')\rho^2} dt', \quad (18)$$

and H_n is the Hermite polynomial of order n . Using the mass and frequency defined previously Eq. (10) reads

$$\ddot{\rho} + (D-1) \frac{\dot{a}}{a} \dot{\rho} + \left[\frac{k^2}{a^2} + \mu^2 + \xi R \right] \rho = \frac{a^{-2(D-3)}(t)}{\rho^3}. \quad (19)$$

To find the exact solutions of Eq. (17), one has to solve Eq.(19) or find the two linearly independent solutions of Eq. (6). Let us consider the latter case. Let $dt = a(t)d\eta$ be the conformal time and let us define a new variable $q_{\lambda ik} = \Omega \bar{q}_{\lambda ik}$. With this, Eq. (6) reads

$$\begin{aligned}\bar{q}'' + \left[2a \frac{\dot{\Omega}}{\Omega} - \dot{a} + (D-1)\dot{a} \right] \bar{q}'_{ik} + \\ [(k^2 + a^2\mu^2 + a^2\xi R) + a^2 \frac{\ddot{\Omega}}{\Omega} + (D-1)a\dot{a} \frac{\dot{\Omega}}{\Omega}] \bar{q}_{ik} = 0\end{aligned}\quad (20)$$

where the prime and the dot means a derivative with respect to η and t respectively. By choosing $\Omega = a^{-(D-1)/2}$ one finds

$$\begin{aligned}\bar{q}''_{\lambda ik} - \dot{a} \bar{q}'_{\lambda ik} + [(k^2 + a^2\mu^2 + a^2\xi R) + \frac{(D-1)(D+1)}{4} \dot{a}^2 \\ - \frac{(D-1)}{2} a\ddot{a} - \frac{(D-1)^2}{2} \dot{a}^2] \bar{q}_{\lambda ik} = 0.\end{aligned}\quad (21)$$

To obtain an exact solution of Eq. (21), let us consider the de Sitter spacetime where $a = e^{Ht}$, and

$$\eta = -\frac{e^{-Ht}}{H} = \frac{1}{Ha(t)}, \quad \dot{a} = -\frac{1}{\eta}, \quad \ddot{a} = -\frac{H}{\eta}. \quad (22)$$

Plugging these relations into Eq. (15) we obtain

$$\bar{q}''_{\lambda ik} + \frac{1}{\eta} \bar{q}'_{\lambda ik} + \left\{ k^2 - \frac{1}{\eta^2} \left[\frac{(D-1)^2}{4} - \frac{a^2\mu^2 + 12H^2\xi}{H^2} \right] \right\} q_{\lambda ik} = 0. \quad (23)$$

which can be written as

$$\left\{ \frac{d^2}{d(k\eta)^2} + \frac{1}{(k\eta)} \frac{d}{d(k\eta)} + \left[1 - \frac{\nu^2}{(k\eta)^2} \right] \right\} \bar{q}_{\lambda ik} = 0. \quad (24)$$

where $R = 12H^2$, and

$$\nu^2 = \frac{(D-1)^2}{4} - \frac{a^2\mu^2 + 12H^2\xi}{H^2}. \quad (25)$$

Equation (24) is a Bessel equation with solutions given by $J_\nu(k|\eta|)$ and $N_\nu(k|\eta|)$. The two linearly independent solutions for q are:

$$q_{\lambda ik} = \begin{cases} a^{\frac{-(D-1)}{2}} J_\nu(k|\eta|) \\ a^{\frac{-(D-1)}{2}} N_\nu(k|\eta|) \end{cases}. \quad (26)$$

Finally, according to Ref.[29, 20], a particular solution of Eq. (19) reads

$$\rho = a^{\frac{-(D-1)}{2}} \left[AJ_\nu^2 + BN_\nu^2 + \left(AB - \frac{\pi^2}{4H^2} \right)^{\frac{1}{2}} J_\nu N_\nu \right]^{\frac{1}{2}}, \quad (27)$$

where A and B are real constants. The fixing of these constants is related to the choice of our vacuum. This is due to the fact that the construction of particle states and the choice of the vacuum is not unique in curved spaces as the one used here. This is important since the production of particles can be inferred only after we choose some vacuum to compare with our physical solution. A natural choice is the Bunch-Davies vacuum, which is the adiabatic vacuum at early times ($t \rightarrow -\infty$)[20]. For this adiabatic vacuum at early times $A = B = \pi/2H$ and ρ becomes

$$\rho = (H\eta)^{\frac{(D-1)}{2}} \sqrt{\frac{\pi}{2H}} [J_\nu^2 + N_\nu^2]^{\frac{1}{2}}, \quad (28)$$

for $a = 1/H\eta$. This is the general solution for the scalar field for arbitrary D . We can see that for $D = 4$ our solution gives

$$\rho = a^{-\frac{3}{2}} \sqrt{\frac{\pi}{2H}} [J_\nu^2 + N_\nu^2]^{\frac{1}{2}}, \quad (29)$$

with

$$\nu^2 = \frac{9}{4} - \frac{a^2\mu^2 + 12H^2\xi}{H^2}. \quad (30)$$

This result agrees with the solution found in Ref. [20] for $D = 4$.

4 Concluding remarks

In this paper we used the Lewis and Riesenfeld method to obtain the time-dependent Schrödinger states emerging from the quantization of the scalar field

in the D -dimensional de Sitter spacetime. There is a similarity between the equations found here and the ones for the electromagnetic field. However, differently from the latter case, we have a parameter ξ that controls the conformality of the system. A general solution for arbitrary D and ξ is therefore very useful to analyze the physics of the problem.

Let us first analyze the $D = 3$ case. This must become important for condensed matter systems. The solution found is identical to that one of the gauge field in $D = 4$ [29] if we fix $\xi = \mu = 0$. One could mistakenly conclude that $\rho = \text{constant}$ is the only solution to the problem. However, we should remember that Eq. (29) is obtained by considering a system evolving to a vacuum state in the limit $t \rightarrow -\infty$ what surely is not the case here. Therefore the solution for $D = 3$ must be given by Eq. (27) and the constants A, B must not be fixed by fundamental arguments but from initial conditions in the referred system. The next case is $D = 4$, and we have seen that our results agree with the one in the literature [20]. Here we can see clearly that the choice of ξ controls the conformality of the system. If we choose $\xi = 1/6$ we obtain a conformal action and the trivial $\rho = \text{constant}$, as expected.

A very intriguing consequence of the results obtained here is for extra dimension physics. This has gained a lot of attention due to superstring theory [1, 2] and Randall-Sundrum models [11, 12]. In such models our universe is conceived as a brane in a five-dimensional space. If we choose a value for ξ to keep the conformal invariance in the brane ($D = 4$), we must lose the conformal invariance from the $D = 5$ viewpoint. A de Sitter spacetime would therefore imply a thermal bath for the comoving referentials in this enlarged space. Therefore, at least in principle, this can add an effective temperature in the membrane that could contribute to the overall dynamics of the universe. However, we should point out that from this viewpoint any field should contribute for this effective temperature and it is not clear for the authors how to separate the contributions.

At last, we would like to point out that the procedure described here can be used to trace the present properties of the quantum scalar field back to the recombination era in an arbitrary D -dimensional universe. This would be a much more interesting phenomenological result.

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